

# Combinatorial Deformations: Twisting and Perturbations

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## Abstract

The framework used to prove the multiplicative law deformation of the algebra of Feynman-Bender diagrams is a *twisted shifted dual law* (in fact, twice). We give here a clear interpretation of its two parameters. The crossing parameter is a deformation of the tensor structure whereas the superposition parameters is a perturbation of the shuffle coproduct of Hoffman type which, in turn, can be interpreted as the diagonal restriction of a superproduct. Here, we systematically detail these constructions.

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# 1 Introduction

In [1], Bender, Brody, and Meister introduced a special field theory, then called “Quantum Field Theory of Partitions”. This theory is based on a bilinear product formula which reads

$$\mathcal{H}(F, G) = F \left( z \frac{d}{dx} \right) G(x) \Big|_{x=0} . \quad (1)$$

If one develops this formula in the case when  $F$  and  $G$  are free exponentials, one obtains a summation over all the (finite) bicoloured graphs with multiple edges and no isolated point [2] (the set of these diagrams will be called **diag**), a data structure which is equivalent to classes of packed matrices [10] under permutations of rows and columns.

So, one has a Feynman-type expansion of the product formula

$$\mathcal{H} \left( \exp \left( \sum_{n=1}^{\infty} L_n \frac{z^n}{n!} \right), \exp \left( \sum_{n=1}^{\infty} V_n \frac{z^n}{n!} \right) \right) = \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\substack{d \in \mathbf{diag} \\ |d|=n}} mult(d) \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)} \quad (2)$$

where  $mult(d)$  is the number of pairs  $(P_1, P_2)$  of (ordered) set partitions of  $\{1, \dots, n\}$  which fit into a diagram  $d$ ,  $|d|$  the number of edges in  $d$  and

$$\mathbb{L}^{\alpha(d)} = L_1^{\alpha_1} L_2^{\alpha_2} \dots ; \quad \mathbb{V}^{\beta(d)} = V_1^{\beta_1} V_2^{\beta_2} \dots \quad (3)$$

is the multiindex notation for the monomials in  $\mathbb{L} \cup \mathbb{V}$  where  $\alpha_i = \alpha_i(d)$  (resp.  $\beta_j = \beta_j(d)$ ) is the number of white (resp. black) spots of degree  $i$  (resp.  $j$ ) in  $d$ .

The set **diag** can receive the structure of a monoid such that the arrow  $d \mapsto \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)}$  is a morphism (of monoids) and then, by linear extension, one deduces a morphism of algebras

$$\mathbb{C}[\mathbf{diag}] \rightarrow \mathbb{C}[\mathbb{L} \cup \mathbb{V}] . \quad (4)$$

For at least three models of Physics, one can specialize  $\mathbb{L}$  so that the canonical Hopf algebra structure of  $\mathbb{C}[\mathbb{L} \cup \mathbb{V}]$  can be lifted, through (4). The resulting Hopf algebra (based on  $\mathbb{C}[\mathbf{diag}]$ ) has been denoted **DIAG**. To our great surprise, this Hopf algebra structure could be lifted at the (noncommutative) level of the objects themselves instead of classes, resulting in the construction of a Hopf algebra on (linear combinations of) “labelled diagrams” (the monoid **ldiag**, see [2]). As these “labelled diagrams” are in one-to-one correspondence with the packed matrices of **MQSym**, we get on the vector space  $\mathbb{C}[\mathbf{ldiag}]$  two (combinatorially natural) structures of algebra (and co-algebra) and one could raise the question of the existence of a continuous deformation between the two. The answer is positive and can be performed through a tree-parameter (two formal, or continuous and one boolean) Hopf deformation<sup>1</sup> of **LDIAG** called **LDIAG**( $q_c, q_s, q_t$ ) [2] such that

$$\mathbf{LDIAG}(0, 0, 0) \simeq \mathbf{LDIAG} ; \quad \mathbf{LDIAG}(1, 1, 1) \simeq \mathbf{MQSym} . \quad (5)$$

The rôle of the two parameters  $q_c, q_s$  (algebra parameters, as  $q_t$  is a coalgebra parameter) was discovered just counting crossings and superpositions in the twisted labelled diagrams

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<sup>1</sup>This algebra deformation has received recently another realisation in terms of bi-words [3].

(see [2] for details). This simple statistics (counting crossings and superpositions) yields an associative law on the diagrams. The first proof given for the associativity was mainly computational and it was a surprise that even the associativity held. In the meanwhile it was evoked by correspondents that the two parameters ( $q_c$  and  $q_s$ ) could be of different nature. The aim of this paper is to answer this question and give a conceptual proof of associativity by developing three building blocks which are separately easy to test: addition of a group-like element to a co-associative coalgebra, codiagonal deformation of a semigroup, extension of a coproduct to words, application of the shifting lemma. The essential ingredient of the two last operations is what has become nowadays a classical tool, the coloured product of algebras for which we give some new results.

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## 1.1 The deformed algebra $\text{LDIAG}(q_c, q_s)$

## 1.2 Review of the construction of the algebra

The complete story of the algebra of Feynman-Bender diagrams which arose in Combinatorial Physics in (2005) can be found in [2] and a fragment of it, as well as a realization with an alternative data structure, in [3].

Recall that (classical) shuffle products (of words) can be expressed in two ways

- a) recursion
- b) summation on (and by means of) some permutations.

Here, we will trace back the construction of the deformed product between two diagrams, starting from an analog of (b) (using however the symmetric semigroup instead of the symmetric group, see below) and going gradually to (a).

The first description of the deformed case was graphic (as it was discovered as such) and goes as follows [2].

- take two labelled diagrams  $d_1, d_2$
- concatenate them. The result, i.e. the diagram obtained in placing  $d_2$  at the right of  $d_1$ , is denoted by  $[d_1|d_2]_L$  [2]
- form the sum

$$[d_1|d_2]_{L(q_c, q_s)} = \sum_{\substack{\text{all crossings and} \\ \text{superpositions of black spots}}} q_c^{nc \times \text{weight}} q_s^{\text{weight} \times \text{weight}} {}_{CS}([d_1|d_2]_L) \quad (6)$$

where

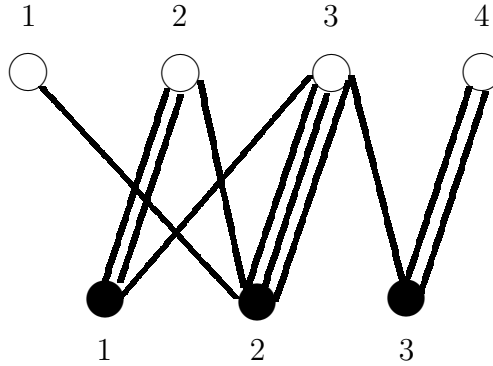
- $q_c, q_s \in K$  (this includes the case when  $q_c, q_s$  are formal as one can always take  $K = k[q]$  or  $k(q)$ )
- the exponent of  $q_c^{nc \times weight}$  is the number of crossings of “what crosses” times its weight
- the exponent of  $q_s^{weight \times weight}$  is the product of the weights of “what is overlapped”
- $cs()$  are the diagrams obtained from  $[d_1|d_2]_L$  by the process of crossing and superposing the black spots of  $d_2$  on to those of  $d_1$ , the order and distinguishability of the black spots of  $d_1$  (resp.  $d_2$ ) being preserved.

This intuitive description can be set rigorously by means of the symmetric semigroup (the symmetric group provides only crossings as for the shuffle product). The symmetric semigroup on  $F$ , a finite set (denoted here  $SSG_F$ ) is the set of endofunctions  $F \rightarrow F$ . In order to preserve the requirement that black spots kept on being labelled from 1 to some integer, we have to ask that the mapping acting on the diagram  $d$  with  $n$  black spots had its image of the type  $[1..m]$  for some  $m \leq n$ . The result noted  $d.f$  has  $m$  black spots such that the black spot of (former) label “ $i$ ” bears the new label “ $f(i)$ ”.

To define this action more formally, we use the description of a diagram  $d$  as a weight function  $\omega : \mathbb{N}^+ \times \mathbb{N}^+ \rightarrow \mathbb{N}$  (as in [3]) such that the supporting subgraph

$$supp(\omega) = \{(i, j) \in \mathbb{N}^+ \times \mathbb{N}^+ \mid w(i, j) \neq 0\} \quad (7)$$

had projections of the form  $pr_1(supp(\omega)) = [1, p]$ ;  $pr_2(supp(\omega)) = [1, q]$  for some  $p, q \in \mathbb{N}^+$ .



**Fig 1.** — *Labelled diagram of format  $3 \times 4$  ( $p = 3$  ;  $q = 4$ ).  
The weight function, when it is not 0, is as follows*

$j$	2	3	1	2	3	3	4
$i$	1	1	2	2	2	3	3
$\omega(i, j)$	2	1	1	1	3	1	2

If we consider an onto mapping  $[1..p] \rightarrow [1..r]$ , the diagram  $\Gamma.f = \Gamma'$  has the following weight function  $\omega'$

$$\omega'(i, k) = \sum_{f(j)=i} \omega(j, k) . \quad (8)$$

Before giving the expression of the deformed product, we must define local partial degrees. For a black spot with label “ $l$ ”, we denote by  $BS(d, l)$  its degree (number of adjacents edges). Then, for  $d_1$  (resp.  $d_2$ ) with  $p$  (resp.  $q$ ) black spots, the product reads

$$[d_1|d_2]_{L(q_c, q_s)} = \sum_{f \in Shs(p, q)} \left( \prod_{\substack{i < j \\ f(i) > f(j)}} q_c^{BS(d, i).BS(d, j)} \right) \left( \prod_{\substack{i < j \\ f(i) = f(j)}} q_s^{BS(d, i).BS(d, j)} \right) [d_1|d_2]_{L.f} \quad (9)$$

where  $Shs(p, q)$  is the set of mappings  $f \in SSG_{[1..p+q]}$  with image of type  $[1 \dots m]$  (with  $\max\{p, q\} \leq m \leq p + q$ ), and such that

$$f(1) < f(2) < \dots < f(p) ; f(p+1) < f(p+2) < \dots < f(p+q) . \quad (10)$$

This condition, similar to that of the shuffle product, guarantees that the black spots of the diagrams are kept in order during the process of shuffling with superposition (hence the name  $Shs$ ).

### 1.3 Coding and the recursive definition

The graphic and symmetric-semigroup-indexed description of the deformed law neither give immediately a recursive definition nor an explanation of “why” the law is associative. We will, on our way to understand this (as well as the different natures of its parameters), proceed in three steps:

- coding the diagrams by words of monomials
- presenting the law as a shifted law
- give a recursive definition of the (non-shifted) law.

The code used here relies on monomials over a commutative alphabet of variables  $\mathbb{X} = \{x_i\}_{i \geq 1}$ . As in [2], we will note  $\mathfrak{MON}(\mathbb{X})$  the monoid of monomials  $\{\mathbb{X}^\alpha\}_{\alpha \in \mathbb{N}(\mathbb{X})}$  (indeed, the free commutative monoid over  $\mathbb{X}$ ) and  $\mathfrak{MON}^+(X) = \{\mathbb{X}^\alpha\}_{\alpha \in \mathbb{N}(\mathbb{X}) - \{0\}}$  the semigroup of its non-unit elements (the free commutative semigroup over  $\mathbb{X}$ ).

Remark that each weight function  $\omega \in \mathbb{N}^{(\mathbb{N}^+ \times \mathbb{N}^+)}$  yields an equivalent “word of monomials”  $W(\omega) = w_1 w_2 \dots w_p$  such that

$$W(\omega)[i] = w_i = \prod_{j=1}^{\infty} x_j^{\omega(i, j)} . \quad (11)$$

The correspondence  $\mathbb{N}^{(\mathbb{N}^+ \times \mathbb{N}^+)} \rightarrow (\mathfrak{MON}^+(X))^*$  is one-to-one and provides at once a way to code each labelled diagram through its weight function as a word of monomials. Conversely a word  $W \in (\mathfrak{MON}^+(X))^*$  is the code of a diagram iff

$$indexes(Alph(W)) = [1..m] \quad (12)$$

(where  $indexes(Alph(W))$  is the set of  $i \in \mathbb{N}^+$  such that an  $x_i$  is involved in  $W$ ). Due to the special indexation of its alphabet, the monoid  $(\mathfrak{MON}^+(X))^*$  comes equipped with a set of endomorphisms, the translations  $T_n$  defined on the variables by  $T_n(x_i) = x_{i+n}$  and

extended to  $\mathfrak{Mon}^+(X)$ , to  $(\mathfrak{Mon}^+(X))^*$  and then to  $K\langle \mathfrak{Mon}^+(X) \rangle$ . For example, the code of a concatenation reads

$$code([d_1|d_2]_L) = code(d_1).T_{max(indexes(Alph(code(d_1)))}(code(d_2)) . \quad (13)$$

The reader may check easily that one can compute recursively the deformed law on the codes by (the function “code” being below extended by linearity)

$$code([d_1|d_2]_L) = code(d_1) \uparrow T_{max(indexes(Alph(code(d_1)))}(code(d_2)) \quad (14)$$

where the bilinear law  $\uparrow$  is recursively defined on the words as follows

$$\begin{cases} 1_{(\mathfrak{Mon}^+(X))^*} \uparrow w &= w \uparrow 1_{(\mathfrak{Mon}^+(X))^*} = w \\ au \uparrow bv &= a(u \uparrow bv) + q_c^{|au||b|} b(au \uparrow v) + q_c^{|u||b|} q_s^{|a||b|} (a \cdot b)(u \uparrow v) \end{cases} \quad (15)$$

where  $(a \cdot b)$  denotes the (monomial) product of  $a$  and  $b$  within  $\mathfrak{Mon}^+(X)$ .

It is this last recursion that we will decompose and analyse below in order to get a better understanding of the parameters.

The associativity of the law (15) is a consequence of the following proposition.

**Proposition 1.1** (*Prop. 5.1 in [2]*) *Let  $(S, \cdot)$  be a semigroup graded by a degree function  $|\cdot| : S \rightarrow \mathbb{N}$  (i. e. a morphism to  $(\mathbb{N}, +)$ ) and  $S^*$  the set of lists (denoted by words  $a_1 a_2 \cdots a_k$ ) with letters in  $S$ .*

*Let  $q_c, q_s \in K$  be two elements in a (commutative) ring  $K$ . We define on  $K\langle S \rangle = K[S^*]$  a new law  $\uparrow$  by*

$$\begin{aligned} w \uparrow 1_{S^*} &= 1_{S^*} \uparrow w = w \\ au \uparrow bv &= a(u \uparrow bv) + q_c^{|au||b|} b(au \uparrow v) + q_c^{|u||b|} q_s^{|a||b|} (a \cdot b)(u \uparrow v) \end{aligned} \quad (16)$$

*where the weights are extended additively to lists (words) by*

$$|a_1 a_2 \cdots a_k| = \sum_{i=1}^k |a_i| .$$

*Then the new law  $\uparrow$  is graded, associative with  $1_{S^*}$  as its unit.*

The questions that have arisen in the introduction can be now reformulated as follows.

Q1) Are  $q_c$  and  $q_s$  of the same nature ?

Q2) If no, can the associativity, beyond direct computation, be explained step by step by constructions which will show their different natures ?

With this end in view, we need to recall a now classical tool, the coloured product of two algebras.

## 2 Colour factors and products

Colours factors were introduced by [13] and the theory was developped or used in [9, 11, 12, 15].

Let  $\mathcal{A} = \oplus_{\alpha \in \mathcal{D}} \mathcal{A}_\alpha$  and  $\mathcal{B} = \oplus_{\beta \in \mathcal{D}} \mathcal{B}_\beta$  be two  $\mathcal{D}$ -graded associative algebras ( $\mathcal{D}$  is a commutative semigroup whose law is denoted additively). Readers that are not familiar with graded algebras can think of  $\mathcal{D} = \mathbb{N}^{(X)}$ , the free commutative monoid over  $X$  and  $\mathcal{A}_\alpha = K[X]_\alpha$ , the space of homogeneous polynomials of multidegree  $\alpha$ .

We suppose given a mapping  $\chi : \mathcal{D} \times \mathcal{D} \longrightarrow K$  and define a law of algebra on  $\mathcal{A} \otimes \mathcal{B}$  by

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = \chi(\beta_1, \alpha_2)(x_1 x_2 \otimes y_1 y_2) \quad (17)$$

for  $(x_i) \in \mathcal{A}_{\alpha_i}$  and  $(y_i) \in \mathcal{B}_{\beta_i}$  ( $i = 1, 2$ ).

The computations of  $((x_1 \otimes y_1)(x_2 \otimes y_2))(x_3 \otimes y_3)$  and  $(x_1 \otimes y_1)((x_2 \otimes y_2)(x_3 \otimes y_3))$  using (17) both lead to the following proposition the second part – converse – of which relies on the existence of free elements.

**Proposition 2.1** [15] *Let  $\chi : \mathcal{D} \times \mathcal{D} \longrightarrow K$ . The following are equivalent*

- i) *For  $\mathcal{A}, \mathcal{B}$   $\mathcal{D}$ -graded associative algebras, the product defined by (17) is associative.*
- ii)  *$(\forall \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \in \mathcal{D}$*

$$\chi(\beta_1, \alpha_2)\chi(\beta_1 + \beta_2, \alpha_3) = \chi(\beta_2, \alpha_3)\chi(\beta_1, \alpha_2 + \alpha_3) \quad (18)$$

**Definition 2.2** *Every mapping  $\chi : \mathcal{D} \times \mathcal{D} \longrightarrow K$  which fulfills the equivalent conditions of proposition (2.1) will be called a colour factor.*

**Remark(s) 2.3** i) *If  $\chi$  is bilinear, which means in this context that the following equations are satisfied (for all  $\alpha, \alpha', \beta, \beta' \in \mathcal{D}$ )*

$$\begin{aligned} \chi(\alpha + \alpha', \beta) &= \chi(\alpha, \beta)\chi(\alpha', \beta) \\ \chi(\alpha, \beta + \beta') &= \chi(\alpha, \beta)\chi(\alpha, \beta') \end{aligned} \quad (19)$$

*then, the two members of (17) amount to*

$$\chi(\beta_1, \alpha_2)\chi(\beta_1, \alpha_3)\chi(\beta_2, \alpha_3) = \prod_{1 \leq i < j \leq 3} \chi(\beta_i, \alpha_j) \quad (20)$$

*and hence  $\chi$  is a colour factor. But the full class of colour factors is much larger than solutions of Eq. (19). Just observe that Eq.(18) is homogeneous in the classical sense i.e. for all  $\lambda \in K$ , if  $\chi$  fulfills (18) then  $\lambda\chi$  still does. Hence, for example, any constant function on  $\mathcal{D} \times \mathcal{D}$  is a colour factor. This shows the existence of colour factors that are not bilinear.*

ii) *It may seem that one could generalize (19) to the case when  $\mathcal{D}$  is noncommutative but, in fact, there is no gain of generality because, as  $K$  is commutative, the bicharacter factorizes through  $\mathcal{D}^{ab}$  (the quotient of  $\mathcal{D}$  by the finest congruence  $\equiv$  such that  $\mathcal{D}/\equiv$  is abelian).*

**Note(s) 2.4** *i) The colour product of two algebras  $\mathcal{A} = \oplus_{\alpha \in \mathcal{D}} \mathcal{A}_\alpha$  and  $\mathcal{B} = \oplus_{\beta \in \mathcal{D}} \mathcal{B}_\beta$  comes also as a graded algebra by*

$$(\mathcal{A} \otimes \mathcal{B})_\gamma = \oplus_{\alpha+\beta=\gamma} \mathcal{A}_\alpha \otimes \mathcal{B}_\beta. \quad (21)$$

*The usual identification*

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \simeq \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) \quad (22)$$

*holds for the coloured products.*

*ii) Moreover, if  $\mathcal{A} \xrightarrow{f} \mathcal{A}'$  (resp.  $\mathcal{B} \xrightarrow{g} \mathcal{B}'$ ) are two morphisms of (graded) algebras, then  $\mathcal{A} \otimes \mathcal{B} \xrightarrow{f \otimes g} \mathcal{A}' \otimes \mathcal{B}'$  is a morphism of algebras (colour products).*

## 3 Special classes of laws

### 3.1 Dual laws

#### 3.1.1 Algebras and coalgebras in duality

An algebra  $(\mathcal{A}, \mu)$  and a coalgebra  $(C, \Delta)$  are called in duality iff there is a non-degenerate pairing  $\langle - | - \rangle$  such that for all  $x, y \in \mathcal{A}$ ,  $z \in C$

$$\langle \mu(x, y) | z \rangle = \langle x \otimes y | \Delta(z) \rangle^{\otimes 2} \quad (23)$$

In the following, we will call *dual law* a law  $K\langle A \rangle \otimes K\langle A \rangle \xrightarrow{*} K\langle A \rangle$  on the free algebra which is the dual of a comultiplication, the pairing being given on the basis of words by  $\langle u | v \rangle = \delta_{u,v}$ .

Our first examples are essential in modern and not-so-modern research ([14, 4]). Firstly, we have the dual of the Cauchy product

$$\Delta_{Cauchy}(w) = \sum_{uv=w} u \otimes v. \quad (24)$$

Contrary to this one (24), which is not a morphism of algebras<sup>2</sup>

$$K\langle A \rangle \longrightarrow K\langle A \rangle \otimes K\langle A \rangle, \quad (25)$$

one has three very well-known examples being so, namely duals of the shuffle  $\sqcup$ , the Hadamard  $\odot$  and the infiltration product  $\uparrow$ . As they are morphisms between the algebras (25), they are well defined by their values on the letters. Respectively

$$\Delta_{\sqcup}(x) = x \otimes 1 + 1 \otimes x; \quad \Delta_{\odot}(x) = x \otimes x; \quad \Delta_{\uparrow}(x) = x \otimes 1 + 1 \otimes x + x \otimes x. \quad (26)$$

One can prove that the deformations  $\Delta_q = \Delta_{\sqcup}(x) + q\Delta_{\odot}(x)$  are also co-associative and that they are the unique solutions of the problem of bialgebra comultiplications on  $K\langle A \rangle$  that are compatible with subalphabets [6].

In the sequel, we will make use several time of the following lemma the proof of which is left to the reader.

**Lemma 3.1** *Let  $\mathcal{A}$  be an algebra and  $\mathcal{C}$  be a coalgebra in (non-degenerate) duality, then  $\mathcal{A}$  is associative iff  $\mathcal{C}$  is coassociative.*

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<sup>2</sup>Unless  $A = \emptyset$ .



### 3.1.2 Duality between grouplike elements and unities

Let  $(\mathcal{C}, \Delta)$  be a coalgebra with counit  $\epsilon$ . We call *group-like* an element  $u$  such that

$$\epsilon(u) = 1 ; \Delta(u) = u \otimes u . \quad (27)$$

One then has  $\mathcal{C} = \ker(\epsilon) \oplus K.u$  and

$$\Delta(y) = \Delta^+(y) + y \otimes u + u \otimes y - \epsilon(y)u \otimes u . \quad (28)$$

where  $\Delta^+$  is a comultiplication on  $\mathcal{C}$  for which  $\ker(\epsilon) = \mathcal{C}^+$  is a subcoalgebra (i. e.  $\Delta^+(\mathcal{C}^+) \subset \mathcal{C}^+ \otimes \mathcal{C}^+$ ) [8].

**Proposition 3.2** *Let  $(\mathcal{C}, \Delta, \epsilon)$  be a coalgebra with counit,  $u$  a group-like element in  $\mathcal{C}$  and  $(\mathcal{C}^+, \Delta^+)$  be as in (28). On the other hand, let  $\mathcal{A}$  be an algebra and  $\mathcal{A}^{(1)} = \mathcal{A} \oplus K.v$  be the algebra with unit constructed from  $\mathcal{A}$  by adjunction of the unity  $v$ . Then, if  $\mathcal{C}^+$  and  $\mathcal{A}$  are in duality by  $\langle | \rangle$ , so are  $\mathcal{C}$  and  $\mathcal{A}^{(1)}$  by  $\langle | \rangle_\bullet$  defined as follows*

$$\langle x + \alpha v | y + \beta u \rangle_\bullet = \langle x | y \rangle + \beta \alpha . \quad (29)$$

*Proof* — Let

$$\begin{aligned} & \langle (x_1 + \alpha_1 v) \otimes (x_2 + \alpha_2 v) | \Delta(y + \beta u) \rangle_\bullet^{\otimes 2} = \\ & \langle (x_1 + \alpha_1 v) \otimes (x_2 + \alpha_2 v) | \Delta^+(y) + y \otimes u + u \otimes y + \beta u \otimes u \rangle_\bullet^{\otimes 2} \end{aligned} \quad (30)$$

but, according to the fact that

$$\langle x_i | u \rangle = \langle x_i \otimes v | \Delta^+(y) \rangle = \langle v \otimes x_i | \Delta^+(y) \rangle = \langle v \otimes v | \Delta^+(y) \rangle = \langle v | y \rangle = 0$$

one has from (30)

$$\begin{aligned} & \langle (x_1 + \alpha_1 v) \otimes (x_2 + \alpha_2 v) | \Delta(y + \beta u) \rangle_\bullet^{\otimes 2} = \\ & \langle x_1 \otimes x_2 | \Delta^+(y) \rangle^{\otimes 2} + \alpha_2 \langle x_1 | y \rangle + \alpha_1 \langle x_2 | y \rangle + \alpha_1 \alpha_2 \beta = \\ & \langle x_1 x_2 + \alpha_2 x_1 + \alpha_1 x_2 + \alpha_1 \alpha_2 v | y + \beta u \rangle_\bullet = \langle (x_1 + \alpha_1 v)(x_2 + \alpha_2 v) | y + \beta u \rangle_\bullet \end{aligned} \quad (31)$$

which proves the claim.  $\square$

## 3.2 Deformed laws

Let  $S$  be a semigroup graded on a semigroup of degrees  $\mathcal{D}$  and  $\mathcal{A} = K[S]$  its algebra. A colour factor  $\chi : \mathcal{D} \times \mathcal{D} \rightarrow K$  being given, we endow the algebra  $\mathcal{A} \otimes \mathcal{A}$  with the coloured tensor product structure. Notice that the diagonal subspace  $D_S = \oplus_{x \in S} Kx \otimes x$  is a subalgebra as

$$(x \otimes x)(y \otimes y) = \chi(|x|, |y|)xy \otimes xy . \quad (32)$$

Carrying (32) back to  $\mathcal{A}$  by means of the isomorphism of vector spaces,  $\mathcal{A} \rightarrow D_S$ , one sees immediately that the deformed product on  $\mathcal{A}$  given by

$$x \cdot_\chi y = \chi(|x|, |y|)xy \quad (33)$$

is associative.

If  $\mathcal{A}$  is endowed with the scalar product for which the basis  $(s)_{s \in S}$  is orthonormal, the pairing is non-degenerate and the dual comultiplication is given by

$$\Delta(z) = \sum_{xy=z} \chi(|x|, |y|)x \otimes y . \quad (34)$$

The construction together with lemma (3.1) proves that this comultiplication on  $\mathcal{A}$  is coassociative.

### 3.3 Shifted laws

We begin by a very general version of the “shifting lemma” (more general than the one given and needed in [2]).

**Lemma 3.3** *Let  $\mathcal{A}$  be an algebra (whose law will be denoted by  $\star$ ) and  $\mathcal{A} = \bigoplus_{\alpha \in \mathcal{D}} \mathcal{A}_\alpha$  a decomposition of  $\mathcal{A}$  as a direct sum over  $\mathcal{D}$ , a semigroup ( $\mathcal{A}$  is then graded but only as a vector space). Let  $\alpha \mapsto T_\alpha: \mathcal{D} \rightarrow \text{End}^{gr}(\mathcal{A})$  be a morphism of semigroups such that  $T_\alpha \in \text{End}^\alpha(\mathcal{A})$ . Explicitely, for all  $\alpha, \beta \in \mathcal{D}$  ;  $x \in \mathcal{A}_\beta$*

$$T_\alpha(x) \in \mathcal{A}_{\alpha+\beta} \text{ and } T_\alpha \circ T_\beta = T_{\alpha+\beta} . \quad (35)$$

*We suppose that the shifted law defined for  $x \in \mathcal{A}_\alpha$  and  $y \in \mathcal{A}$  by*

$$x \bar{\star} y = x \star T_\alpha(y) \quad (36)$$

*is graded for the decomposition  $\mathcal{A} = \bigoplus_{\alpha \in \mathcal{D}} \mathcal{A}_\alpha$ .  
Then, if the law  $\star$  is associative so is the law  $\bar{\star}$ .*

*Proof* — One has just to prove the identity of associativity of  $\bar{\star}$  for homogeneous elements. Suppose that  $\star$  is associative, for  $x \in \mathcal{A}_\alpha$ ,  $y \in \mathcal{A}_\beta$  and  $z \in \mathcal{A}$ , one has

$$\begin{aligned} x \bar{\star} (y \bar{\star} z) &= x \star T_\alpha(y \bar{\star} z) = x \star T_\alpha(y \star T_\beta(z)) = x \star (T_\alpha(y) \star T_\alpha(T_\beta(z))) = \\ &= x \star (T_\alpha(y) \star T_{\alpha+\beta}(z)) = (x \star T_\alpha(y)) \star T_{\alpha+\beta}(z) = \underbrace{(x \bar{\star} y)}_{\text{of degree } \alpha+\beta} \star T_{\alpha+\beta}(z) = (x \bar{\star} y) \bar{\star} z \end{aligned}$$

□

## 4 Application to the structure of $LDIAG(q_c, q_s)$

### 4.1 Associativity of $LDIAG(q_c, q_s)$ using the previous tools

As was stated in paragraph (1.3), we just have to prove proposition (1.1) and we keep the notations of it. We first remark, from paragraph (3.2) that, for a semigroup  $S$  of type (D)<sup>3</sup>, graded by a degree function  $|\cdot| : S \rightarrow \mathbb{N}$ , the comultiplication  $\Delta_1 : K[S] \rightarrow K[S] \otimes K[S]$  given for  $s \in S$  by

$$\Delta_1(s) = \sum_{rt=s} q_s^{|r||t|} r \otimes t \quad (37)$$

is coassociative.

Now, we endow  $K\langle S \rangle \otimes K\langle S \rangle$  with the structure of coloured product given by the bicharacter on  $S^*$

$$\chi(u, v) = \prod_{\substack{1 \leq i \leq |u| \\ 1 \leq j \leq |v|}} q_c^{[u[i]][v[j]]} \quad (38)$$

One defines a mapping  $\Delta : S \rightarrow K\langle S \rangle \otimes K\langle S \rangle$  by

$$\Delta(s) = s \otimes 1_{S^*} + 1_{S^*} \otimes s + \Delta_1(s) \quad (39)$$

---

<sup>3</sup>After [7], a semigroup  $S$  of type (D) is such that the product mapping  $S \times S \rightarrow S$  has finite fibers.

which is extended at once as a morphism of algebras  $\Delta : K\langle S \rangle \rightarrow K\langle S \rangle \otimes K\langle S \rangle$ . Note that  $V = \oplus_{x \in S \cup \{1_{S^*}\}} K.x = KS \oplus K.1_{S^*}$  is a subcoalgebra for  $\Delta$  and the coalgebra  $V$  is, by paragraph (3.1.2), still coassociative. Now, one has to prove that the following rectangle is commutative

$$\begin{array}{ccc} K\langle S \rangle & \xrightarrow{\Delta} & K\langle S \rangle \otimes K\langle S \rangle \\ \Delta \downarrow & & \downarrow Id \otimes \Delta \\ K\langle S \rangle \otimes K\langle S \rangle & \xrightarrow{\Delta \otimes Id} & K\langle S \rangle \otimes K\langle S \rangle \otimes K\langle S \rangle \end{array} \quad . \quad (40)$$

By Note (2.4) (ii) all the arrows are morphisms of algebras and in particular the composites  $(Id \otimes \Delta) \circ \Delta$  ;  $(\Delta \otimes Id) \circ \Delta$  which, it has been proved just previously, coincide on  $S$  (coassociativity of the subcoalgebra  $V$ ). This shows that the rectangle (40) is commutative.

**End of the setting.** —

We denote by  $\downarrow$  the law which is dual to  $\Delta$ . This law, being dual to a coassociative comultiplication, is associative. We prove that it satisfies the same recursion as in proposition (1.1) so,  $\downarrow = \uparrow$ . It is sufficient to prove the recursion for non-empty factors. One has

$$\begin{aligned} au \downarrow bv &= \sum_{w \in S^*} \langle au \downarrow bv | w \rangle w = \sum_{w \in S^+} \langle au \downarrow bv | w \rangle w = \sum_{\substack{x \in S \\ w_1 \in S^*}} \langle au \downarrow bv | xw_1 \rangle xw_1 = \\ &= \sum_{\substack{x \in S \\ w_1 \in S^*}} \langle au \otimes bv | \Delta(x) \Delta(w_1) \rangle xw_1 = \\ &= \sum_{\substack{x \in S \\ w_1 \in S^*}} \langle au \otimes bv | (x \otimes 1 + 1 \otimes x + \sum_{yz=x} \chi(|y|, |z|) y \otimes z) \Delta(w_1) \rangle xw_1 = \\ &= \sum_{\substack{x \in S \\ w_1 \in S^*}} \langle au \otimes bv | (x \otimes 1) \Delta(w_1) \rangle xw_1 + \sum_{\substack{x \in S \\ w_1 \in S^*}} \langle au \otimes bv | (1 \otimes x) \Delta(w_1) \rangle xw_1 + \\ &= \sum_{\substack{x \in S \\ w_1 \in S^*}} \langle au \otimes bv | (\sum_{yz=x} \chi(|y|, |z|) y \otimes z) \Delta(w_1) \rangle xw_1 = \\ &= \sum_{\substack{x=a \\ w_1 \in S^*}} \langle au \otimes bv | (x \otimes 1) \Delta(w_1) \rangle xw_1 + \sum_{\substack{x \in S \\ w_1 \in S^*}} \langle au \otimes bv | (1 \otimes x) \sum_{i,j} \beta_{ij} w_i \otimes w_j \rangle xw_1 + \\ &= \sum_{\substack{x \in S \\ w_1 \in S^*}} \langle au \otimes bv | (\sum_{yz=x} \chi(|y|, |z|) y \otimes z) \sum_{i,j} \beta_{ij} w_i \otimes w_j \rangle xw_1 = \\ &= a(u \downarrow bv) + q_c^{|au||b|} b(au \downarrow v) + q_c^{|u||b|} q_s^{|a||b|} (a.b)(u \downarrow v) \end{aligned} \quad (41)$$

which proves the claim.

## 4.2 Structure of $LDIAG(q_c, q_s)$

This section is devoted to the thorough study of the structure of  $\mathbf{LDIAG}(q_c, q_s)$  using that of the algebra of  $(\mathfrak{M}\mathfrak{D}\mathfrak{N}^+(X))^*$  endowed with the shifted law  $\bar{\uparrow}$ .

We first investigate the structure of the monoid  $((\mathfrak{M}\mathfrak{D}\mathfrak{N}^+(X))^*, \bar{\star})$  broadening out to some extent Proposition 3.1 of [2]. For a general monoid,  $(M, \star, 1_M)$ , the irreducible

elements are the elements  $x \neq 1_M$  such that  $x = y \star z \implies 1_M \in \{y, z\}$ . The set of these elements will be denoted  $\text{irr}(M)$ . For convenience, in the following statement,  $M$  stands for the monoid  $((\mathfrak{M}\mathfrak{D}\mathfrak{N}^+(X))^*, \bar{\star})$ ,  $M^+ = M - \{1_M\}$  and  $M_c$  is the submonoid of codes of diagrams (i. e. words which fulfill Eq. (12)).

The monoid  $M$  is free. An element  $w = m_1.m_2.\dots.m_l \in M^+$  (hence  $l = |w| > 0$ ) is reducible iff there exists  $0 < k < l$  such that

$$\left( \text{indices}(\text{Alph}(m_1.m_2.\dots.m_k)) \right) \prec \left( \text{indices}(\text{Alph}(m_{k+1}.m_{k+2}.\dots.m_l)) \right) \quad (42)$$

where, for two nonempty subsets  $X, Y \subset \mathbb{N}^+$ , one notes  $\prec$  the relation of majoration i. e.

$$(\forall (x, y) \in X \times Y)(x < y) \quad (43)$$

One checks at once that the monoid  $M_c$  is generated by  $\text{irr}(M) \cap M_c$  and therefore is free. Now, we need a classical lemma of general algebra.

Let  $(\mathcal{A}, \mu)$  be an algebra endowed with an increasing exhaustive filtration  $(\mathcal{A}_n)_{n \in \mathbb{N}}$  (i. e. two-sided ideals such that  $\mathcal{A}_n \subset \mathcal{A}_{n+1}$  and  $\cup_{n \in \mathbb{N}} \mathcal{A}_n = \mathcal{A}$ ). It is classical to construct the associated graded algebra  $\text{Gr}(\mathcal{A}) = \oplus_{n \geq 0} \mathcal{A}_n / \mathcal{A}_{n-1}$  by passing the law to quotients i. e.  $\bar{\mu}_{p,q} : \mathcal{A}_p / \mathcal{A}_{p-1} \otimes \mathcal{A}_q / \mathcal{A}_{q-1} \rightarrow \mathcal{A}_{p+q} / \mathcal{A}_{p+q-1}$  (one sets  $\mathcal{A}_{-1} = \{0\}$ ). This classical lemma states that, if the associated graded algebra is free, so is  $\mathcal{A}$ .

Now, returning to  $(K\langle \mathfrak{M}\mathfrak{D}\mathfrak{N}^+(X) \rangle, \bar{\uparrow})$  ( $\bar{\uparrow}$  is the shifted deformed law), one constructs a filtration by the number of irreducible components of a word of monomials (call it  $l(w)$  for  $w \in \mathfrak{M}\mathfrak{D}\mathfrak{N}^+(X)$ ). From (16), one gets,

$$w_1 \bar{\uparrow} w_2 = w_1 \bar{\star} w_2 + \sum_{l(w) < l(w_1) + l(w_2)} P_w(q_c, q_s) w \quad (44)$$

with  $P_w \in K[q_c, q_s]$  and then, the associated graded algebra is, by definition, free. One can then state the following structure theorem.

**Theorem 4.1** *The algebra  $K\langle \mathfrak{M}\mathfrak{D}\mathfrak{N}^+(X) \rangle$ , endowed with the shifted deformed law  $\bar{\uparrow}$  is free on the irreducible words and then the algebra  $\mathbf{LDIAG}(q_c, q_s)$  isomorphic to a subalgebra generated by irreducible words is free for every choice of  $(q_c, q_s)$ .*

## 5 Conclusion

To sum up quickly, what has been done in this paper, we can state that the deformed algebra  $\mathbf{LDIAG}(q_c, q_s)$  which originates from a special quantum field theory [1] is free and its law can be constructed from very general procedures: it is a shifted twisted law. Before shifting, one can observe that the law is, in fact, dual to a comultiplication on a free algebra. This comultiplication is a perturbation, with  $q_s$  (the superposition parameter) of the shuffle comultiplication on this free algebra. the extension to words of the comultiplication is done by means of a coloured product operated with  $q_c$  (the crossing parameter). So the two parameters are of very different nature  $q_s$  being a perturbation of the shuffle coproduct and  $q_c$  a deformation of the tensor structure.

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